Photographing the wave function of the Universe

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Abstract

We show that density fluctuations in standard inflationary scenarios may take the most general non-Gaussian distribution if the wave function of the Universe is not in the ground state. We adopt the Schrödinger picture to find a remarkable similarity between the most general inflaton wavefunction and the Edgeworth expansion used in probability theory. Hence we arrive at an explicit relation between the cumulants of the density fluctuations and the amplitudes or occupation numbers of the various energy eigenstates. For incoherent superpositions only even cumulants may be non-zero, but coherent superpositions may generate non-zero odd cumulants as well. Within this framework measurements of cumulants in Galaxy surveys directly map the wavefunction of the Universe.

1 Introduction

Gaussian distributed density fluctuations are seen as a hallmark of the inflationary paradigm [1–4]. In addition, Gaussianity greatly simplifies calculations as only one set of parameters, the power spectrum, needs to be calculated. For these reasons, Gaussianity is commonly assumed as a matter of course. However in non-minimal models of inflation [5–7] it is possible to generate non-Gaussian statistics. The magnitude [8] and type of departures from Gaussianity in these models appear to be restricted; in the simplest cases fluctuations are the square, or a power, of a Gaussian random field [7]. Furthermore, there have been a number of recent claims of detection of a non-Gaussian signal in the COBE data [9–15]. It is therefore of interest to explore further non-Gaussian inflationary models. For instance, one may wonder whether inflation can ever produce fluctuations with the most general probability distribution. It could be the case that some types of non Gaussianity are barred from even the most contrived inflationary models.

Preprint submitted to Elsevier Preprint

In this paper we consider standard inflationary models, and revisit the derivation of the Gaussianity of density perturbations in these models. Microphysical fluctuations in the inflaton field then satisfy very approximately a linear harmonic oscillator equation. These should be subject to quantisation, with the common assumption being that the field lies in the ground state, which for an harmonic oscillator has a Gaussian wave function (eg. [16]). These microphysical fluctuations are then stretched by inflationary expansion, reaching what we nowadays call cosmological length scales. The square of their wavefunction provides their probability distribution, and since the square of a Gaussian is a Gaussian, these fluctuations have Gaussian statistics [1] .

Let us now assume that inflation is triggered by quantum cosmology. However let us assume nothing from quantum cosmology (cf. [17,18]) about the initial form of the inflaton's wavefunction, ψ, allowing it to be given by the most general superposition of energy eigenstates (a situation already considered in [19,20]). The wavefunctions ψ_n of these eigenstates are generally a Gaussian multiplied by a Hermite polynomial H_n [16]. The square of the wave function, $|\psi|^2$, is the probability distribution of fluctuations emerging from such a scenario. The distribution we have just found has a striking resemblance to the Edgeworth expansion used in probability theory [21]. The Edgeworth expansion parameterises the most general distribution, and takes the form of a Gaussian multiplied by a series of Hermite polynomials. The coefficients in this series are polynomials in the cumulants of the distribution. Hence we may write a simple expression relating the components of ψ and the cumulants of the density fluctuations' distribution. The purpose of this Letter is to establish this interesting connection between observational and quantum cosmology.

The plan of this Letter is as follows. In section 2 we review inflationary fluctuations, purposefully adopting the Schrödinger picture. In section 3 we review the other tool used in this Letter – the Edgeworth expansion. Then in sections 4 and 5 we establish the connection between the amplitudes of the various modes in the wavefunction and the cumulants of the distribution of the ensuing density fluctuations. We find that for an incoherent superposition of eigenmodes one may only obtain non-Gaussian distributions with non-zero even cumulants. However for coherent superpositions it is possible to generate the most general non-Gaussian distribution. Our results therefore show that if we take standard inflation for granted, we can in principle experimentally map the wavefunction of the Universe. The implications of such a measurement are briefly discussed in a closing section.

[1] Contrary to a popular myth, the Gaussianity of standard inflation's fluctuations is not due to the central limit theorem.

2 Inflationary fluctuations in the Schrödinger picture

We first review the formalism used for describing inflationary fluctuations in the Schrödinger picture (eg. [19,22]). We assume that inflation arises from a single scalar field, the inflaton field ϕ. We write ϕ as an homogeneous background $\bar{\phi}$ plus a perturbation $\delta\phi$:

$$\phi(\mathbf{x}, t) = \bar{\phi}(t) + \delta\phi(\mathbf{x}, t) \tag{1}$$

with $\delta\phi \ll \bar{\phi}$. It is standard to decompose the perturbation $\delta\phi(\mathbf{x}, t)$ into Fourier modes $\phi_{\mathbf{k}}$. In terms of these, the energy density fluctuations are given by [19,22]:

$$\delta\rho(\mathbf{k}, t) = \dot{\bar{\phi}}(t)\dot{\phi}_{\mathbf{k}}(t) \tag{2}$$

Density fluctuations are therefore proportional to $\dot{\phi}_{\mathbf{k}}$, which in turn is proportional to $\phi_{\mathbf{k}}$. Therefore deviations from Gaussianity in $\delta\rho$ may be studied directly in terms of the statistics of $\phi_{\mathbf{k}}$ (though this is not true in the models considered in [23]).

At early times we can adopt the free field theory limit and the modes $\phi_{\mathbf{k}}$ are then completely decoupled and behave as individual harmonic oscillators with a time-dependent mass $m(t)$ [19,22]. The form of $m(t)$ is dictated by the form of the potential $V(\phi)$ driving the inflaton field. In the slow-roll limit, and for the relevant wavenumbers, the mass $m(t)$ is very approximately constant. We can therefore use the standard quantum mechanical treatment for the harmonic oscillator [16] to find the wave function for $\phi_{\mathbf{k}}$ (hereafter abbreviated as $\psi \equiv \psi(\phi_{\mathbf{k}})$).

In general ψ may be written as a superposition of energy eigenmodes:

$$\psi = \sum \alpha_n \psi_n \tag{3}$$

where n labels the energy spectrum $E = \hbar\omega(n + 1/2)$. The ψ_n take the form

$$\psi_n(\phi) = C_n H_n\left(\frac{\phi}{\sqrt{2}\sigma_0}\right) e^{-\frac{\phi^2}{4\sigma_0^2}} \tag{4}$$

with normalisation fixing C_n as,

$$C_n = \frac{1}{(2^n n! \sqrt{2\pi}\sigma_0)^{1/2}} \,. \tag{5}$$

Here σ_0^2 is the variance associated with the (Gaussian) probability distribution for the ground state $|\psi_0|^2$. In other words σ_0^2 is the power spectrum as derived in standard calculations in which the field is assumed to be in the ground state. σ_0 is therefore related to the potential $V(\phi)$ and to m according to the standard formulae [19,22]. We shall work with Hermite polynomials $H_n(x)$ defined as

$$H_n(x) = (-1)^n e^{x^2} \frac{d^x}{dx^n} e^{-x^2} \tag{6}$$

and normalised as

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx = 2^n \pi^{1/2} n! \delta_{nm}. \tag{7}$$

The most general probability density for the fluctuations in ϕ is thus $P = |\psi|^2$, with ψ given by (3). The ground state (or "zero-point") fluctuations are Gaussian, but any admixture with other states will be reflected in a non-Gaussian distribution function [19,20].

It is worth explaining the apparent discrepancy between our expressions and those in [20]. In [20] the authors regard each mode ϕ_k as a complex variable, and find a wave function for its modulus and phase, such that the probability distribution is most naturally expanded in Laguerre polynomials. In contrast, we consider the real and imaginary parts of ϕ_k as independent real variables, and find separate wavefunctions for each. In this way we find an expansion in Hermite polynomials, allowing for a simpler connection with the Edgeworth expansion. The two treatments are equivalent and the probability densities obtained in [20] can be re-expressed in terms of Hermite polynomials, and vice versa. The results in this Letter still apply, although the algebra is significantly more tedious.

3 The Edgeworth expansion

We now review the Edgeworth expansion [21]. This expansion is a particular form of a class of series known as the Gram-Charlier Type A series. These are used to construct a distribution $P(x)$ by means of its n-th order moments μ_n or cumulants κ_n. These series are convergent under general conditions for $P(x)$ [21,24].

For a standardized variable x (ie $\langle x\rangle = 0$ and $\langle x^2\rangle = 1$) the Edgeworth expan-

sion takes the form

$$P(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \sum_n c_n \overline{H}_n(x)\,. \tag{8}$$

If x is not standartized this expression can be easily modified. The coefficients c_n are tabulated polynomials in the moments μ_i or in the cumulants κ_i. Unfortunately the large body of complex formulae available in the literature uses Hermite polynomials $\overline{H}_n$ defined as

$$\overline{H}_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2} \tag{9}$$

as opposed to $H_n(x)$ defined in (6). The two conventions may be bridged by means of [25]

$$\overline{H}_n(x) = \frac{H_n(x/\sqrt{2})}{2^{n/2}} \tag{10}$$

If $P(x)$ is expanded around a Gaussian with the correct variance $\sigma^2 = \langle x^2 \rangle$ (with $\langle x \rangle = 0$) then the Edgeworth expansion is:

$$\begin{aligned} P(x) = & \frac{e^{\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma^2}\Big(1 + \frac{\kappa_3}{12\sqrt{2}} H_3\left(\frac{x}{\sqrt{2}\sigma}\right) + \frac{\kappa_4}{96} H_4\left(\frac{x}{\sqrt{2}\sigma}\right) + \frac{\kappa_5}{480\sqrt{2}} H_5\left(\frac{x}{\sqrt{2}\sigma}\right) \\ & + \frac{\kappa_6 + 10\kappa_3^2}{4960} H_6\left(\frac{x}{\sqrt{2}\sigma}\right) + ...\Big) \end{aligned} \tag{11}$$

It is also possible to expand $P(x)$ around a Gaussian with a different variance and mean. The coefficients in the expansion are then more complicated, but are listed in [21].

4 Coherent States

Let us initially consider a *coherent* superposition of form (3). We first assume mild non-Gaussianity which we define through the condition $|\alpha_0|^2 \gg |\alpha_i|^2$, for $i \geq 1$. We can then consider the expansion of $|\psi|^2$ only to first order in α_i,

$$P(\phi) = |\psi|^2 = \frac{e^{-\frac{\phi^2}{2\sigma_0^2}}}{\sqrt{2\pi}\sigma_0} \left[1 + \sum_{n\geq 1} \frac{2\Re(\alpha_n)}{(2^n n!)^{1/2}} H_n\left(\frac{\phi}{\sqrt{2}\sigma_0}\right)\right] \tag{12}$$

where we have taken α_0 to have zero phase (so that to first order $\alpha_0 = 1$).

This leads to a one-to-one correspondence between the amplitudes of the various energy eigenstates, and the combinations of cumulants appearing as coefficients in the Edgeworth expansion. The latter simplify enormously if we only keep first order terms: we find that $\kappa_n \propto \Re(\alpha_n)$. Hence to first order the coherent contamination of the ground state by the n^{th}energy eigenstate is signalled by a non-vanishing cumulant κ_n. For instance the presence of the third energy eigenstate results in $\kappa_3 \propto \Re(\alpha_3) \neq 0$, and, to first order, zero higher order cumulants.

Of course we do not need to assume that $|\alpha_0|^2 \gg |\alpha_i|^2$ for $i \geq 1$, but then the series' convergence will have to be more carefully watched. For completeness, and for later use, we consider the more general case as well, making no assumptions about the relative amplitudes of the α_i's. We then have that,

$$P(\phi) = |\psi|^2 = \frac{e^{-\frac{\phi^2}{2\sigma_0^2}}}{\sqrt{2\pi}\sigma_0} \sum_{i,j} \frac{\alpha_i^* \alpha_j}{(2^{i+j} i! j!)^{1/2}} H_i\left(\frac{\phi}{\sqrt{2}\sigma_0}\right) H_j\left(\frac{\phi}{\sqrt{2}\sigma_0}\right) \tag{13}$$

We may recover the Edgeworth expansion by noting that,

$$e^{-x^2} H_i(x) H_j(x) = e^{-x^2} \left[\sum_n b_n^{ij} H_n(x)\right] \tag{14}$$

with,

$$b_n^{ij} = \frac{2^{s-n} i! j!}{(s-n)!(s-i)!(s-j)!} \tag{15}$$

with $2s = n + i + j$. This expression is subject to two selection rules: $n + i + j$ must be even, and the triangular inequality $|i - j| < n < i + j$. One may derive (15) using (7) and the standard result for the integral over a product of three Hermite polynomials (formula 7.375.2 of [26]). Thus we obtain the more complicated expression

$$P(\phi) = |\psi|^2 = \frac{e^{-\frac{\phi^2}{2\sigma_0^2}}}{\sqrt{2\pi}\sigma_0} \sum_n \left(\sum_{i,j} \frac{b_n^{ij} \alpha_i^* \alpha_j}{(2^{i+j} i! j!)^{1/2}}\right) H_n\left(\frac{\phi}{\sqrt{2}\sigma_0}\right) \tag{16}$$

Hence the coherent admixture of the $i^{th}$and j^{th}eigenstates leads to terms in the Edgeworth expansion to all orders between $|i-j|$ and $i+j$. Note that in general one has $\langle\phi\rangle \neq 0$, and that the variance of ϕ may be different from σ_0, requiring suitable modifications to the coefficients in the Edgeworth expansion.

5 Incoherent States

If the various energy eigenstates are added incoherently we obtain the probability distribution,

$$P(\phi) = |\psi|^2 = \sum_n |\alpha_n|^2 \psi_n^2 = \frac{e^{-\frac{\phi^2}{2\sigma_0^2}}}{\sqrt{2\pi}\sigma_0} \sum_n \frac{|\alpha_n|^2}{2^n n!} H_n^2\left(\frac{\phi}{\sqrt{2}\sigma_0}\right) \tag{17}$$

As in the previous section this may be rewritten as

$$P(\phi) = \frac{e^{-\frac{\phi^2}{2\sigma_0^2}}}{\sqrt{2\pi}\sigma_0} \sum_k \left(\sum_n \frac{b_k^{nn}|\alpha_n|^2}{2^n n!}\right) H_k\left(\frac{\phi}{\sqrt{2}\sigma_0}\right) \tag{18}$$

and one may now use the Edgeworth expansion results to infer the cumulants of the distribution. We see that in this case $\langle\phi\rangle = 0$. However the variance of the perturbed distribution is never the same as σ_0^2 and is given by,

$$\sigma^2 = \sigma_0^2 \sum_n |\alpha_n|^2 (2n+1) \tag{19}$$

a result which can be read off from (18) (with the aid of [21]), or easily derived in the Heisenberg picture. Hence formula (19), providing $\sigma^2 - \sigma_0^2$, must be used when solving for the cumulants of the distribution from the coefficients in (18).

The series (18) only contains Hermite polynomials of even order, since b_k^{nn} can only be non-zero for even k. Hence, we note the important result that incoherent superpositions generate non-Gaussian distributions with zero odd cumulants, in contrast with what was obtained in the previous section (as also mentioned in [19]). Furthermore for incoherent superpositions the presence of a non-vanishing n^{th}cumulant signals the presence of the $n/2^{th}$energy eigenstate, in contrast with the result for coherent superpositions. In general, even to lowest order (which is second order in the amplitudes) one may not establish a one-to-one relation between cumulants κ_n and the $|\alpha_n|^2$. The incoherent contamination of the ground state by the n^{th}energy eigenstate generally leads to non-vanishing even cumulants of all orders up to $2n$.

6 Discussion

We have shown that standard inflation may generate density fluctuations with the most general one-point distribution function, if we allow the inflaton field

not to be in the ground state. According to this scenario the cumulants of the distribution provide a measurement of the components of the wave function of the inflaton field, in terms of energy eigenstates. The explicit expression depends on whether the superposition is coherent or incoherent. In the coherent case all cumulants can be potentially non-zero. For incoherent superpositions only non-zero even cumulants can arise.

There is however one type of non-Gaussianity which cannot be generated in this scenario: non-Gaussian inter-mode correlations. Translational invariance requires that the two-point function for modes $\phi_{\mathbf{k}}$ be diagonal ($\langle\phi^\star_{\mathbf{k}}\phi_{\mathbf{k}'}\rangle = P_\phi(k)\delta(\mathbf{k}-\mathbf{k}')$). If the $\phi_{\mathbf{k}}$ are Gaussian they are fully determined by their two-point function, and so translational invariance requires them to be independent random variables. However if the reduced higher order correlators *did* indeed exist they would need not be diagonal as a result of translational invariance. For instance $\langle\phi^\star_{\mathbf{k}}\phi'_{\mathbf{k}}\phi''_{\mathbf{k}}\rangle$ may be non-vanishing for all modes satisfying $\mathbf{k} = \mathbf{k}' + \mathbf{k}''$. Hence non-Gaussianity allows inter-mode correlations, even if we impose translational invariance. These correlation cannot be generated in the scenario we have proposed, because the modes $\phi_{\mathbf{k}}$ are genuine independent random variables.

One may wonder whether the inflaton states we have considered are consistent with inflation. In most of the literature on inflationary density fluctuations the inflaton is assumed to be in the ground state; two notable exceptions being [19] and [20]. This simplification has been justified using, amongst others, the rather dubious rationale that the field must be maximally symmetric, i.e. no scale should be privileged. However scenarios have been considered where a preferred scale is present, in particular that the Planck scale itself is a preferred scale [19]. A more physical argument for taking the ground state has also been proposed: that there must be a cutoff in the energy spectrum of the resulting inflaton particles to avoid infinite energy densities [27]. The cutoff must also be low enough so that the energy density of the inflaton particles does not dominate over the potential driving inflation. The value of this cutoff is uncomfortably low if the inflaton particles were ever in thermal equilibrium. However it could happen that inflation is not preceded by a thermal phase, but starts straight out of quantum cosmology. In such a scenario one should not rule out mild admixtures of states above the ground state, with "mild" being defined in the sense that any backreaction must be negligible.

It is of course debatable whether or not such superpositions of states can be justified by quantum cosmology. The Hartle and Hawking no-boundary proposal [17,18] and the Vilenkin tunneling wave proposal [28] imply that the inflaton wave function must be in the ground state. It would be interesting to investigate the implications to quantum cosmology of a detection of non-Gaussianity.

Acknowledgments

We thank Andy Albrecht, Pedro Ferreira, and Jonathan Halliwell for useful comments. We acknowledge financial support from the Beit Fellowship for Scientific Research (CRC), PPARC (RB) and the Royal Society (JM).

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